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Two-point correlation functions in perturbed minimal models

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Abstract. Two-point correlation functions of the off-critical primary fields $\phi_{1,1+s}$ are considered in the perturbed minimal models $M_{2,2N+3}+\phi_{1,3}$. They are given as infinite series of form factor contributions. The form factors of $\phi_{1,1+s}$ are conjectured from the known results for those of $\phi_{1,2}$ and $\phi_{1,3}$. The conjectured form factors are rewritten in a form which is convenient for summation.

1. Introduction

Correlation functions are important tools to study quantum field theories. In many two-dimensional models, it is known that the determinant representation is useful for non-perturbative analysis of correlation functions [1–11].

In a class of $(1+1)$-dimensional, massive, integrable models [12–25], correlation functions of some operators can be written as an infinite sum over intermediate states and are analysed through the form factor bootstrap procedure [12, 13].

Recently, it has been shown that determinant representation of integral operators is useful to sum up the infinite series in the sinh-Gordon model [26] and in the scaling Lee–Yang model [27]. In these models, an auxiliary Fock space and auxiliary Bose fields, which are called dual fields, are introduced. This approach was developed in [5, 28, 29]

The scaling Lee–Yang model [30] can be identified with the $N=1$ case of the perturbed minimal model $M_{2,2N+3}+\phi_{1,3}$ [31]. The purpose of this paper is to generalize the result of [27] to arbitrary N and to show that the determinant representation is useful also in the perturbed minimal conformal field theories.

The minimal model $M_{2,2N+3}$ is non-unitary and contains $N+1$ scalar primary fields $\phi_{1,1+s}=\phi_{1,2N+2-s}$ $(s=0,\ldots,2N+1)$ with scaling dimensions $(\Delta_{(1,1+s)},\Delta_{(1,1+s)})$ [32]:

$$\Delta_{(1,1+s)}=-\frac{s(2N+1-s)}{2(2N+3)}\qquad s=0,\ldots,2N+1. \tag{1.1}$$

The primary operator $\phi_{1,1}=\phi_{1,2N+2}$ is the identity operator.

The $\phi_{1,3}$-perturbation of $M_{2,2N+3}$ is known to be integrable and is described by the $A^{(2)}_{2N}$-type factorizable scattering theory. The mass spectrum of $A^{(2)}_{2N}$ theory consists of N scalar particles with mass

$$m_a=2m\sin(a\pi/h)\qquad a=1,\ldots,N \tag{1.2}$$

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where $h = 2N + 1$ is the Coxeter number of the Lie algebra $A_{2N}^{(2)}$. The two-body scattering amplitude is given by [33]

$$S_{ab}(\beta) = \prod_{\substack{x=|a-b|+1 \\ \text{step } 2}}^{a+b-1} \{x\}_{(\beta)} \tag{1.3}$$

where

$$\{x\}_{(\beta)} = \frac{\tanh\frac{1}{2}(\beta + (x-1)\pi\mathrm{i}/h)\tanh\frac{1}{2}(\beta + (x+1)\pi\mathrm{i}/h)}{\tanh\frac{1}{2}(\beta - (x-1)\pi\mathrm{i}/h)\tanh\frac{1}{2}(\beta - (x+1)\pi\mathrm{i}/h)}. \tag{1.4}$$

It is conjectured that the conformal primary fields $\phi_{1,1+s}$ become off-critical primary fields [19]. We use the same symbol $\phi_{1,1+s}$ to denote the corresponding off-critical primary operators.

Form factors of a local operator $\mathcal{O}(x)$ are defined as the matrix elements between the vacuum state $\langle\mathrm{vac}\,|$ and n particle states characterized by particle species a_i ($a_i \in \{1, \ldots, N\}$) and rapidities β_i ($i = 1, \ldots, n$):

$$F^{\mathcal{O}}_{a_1\ldots a_n}(\beta_1, \ldots, \beta_n) = \langle\mathrm{vac}\,|\mathcal{O}(0)|\beta_1, \ldots, \beta_n\rangle_{a_1\ldots a_n}. \tag{1.5}$$

The multiparticle form factors for $\phi_{1,2}$ and $\phi_{1,3}$ were calculated in [14]

$$F^{\phi_{1,2}}_{a_1\ldots a_n}(\beta_1, \ldots, \beta_n) = f_{0;a_1\ldots a_n}(\beta_1, \ldots, \beta_n)\prod_{j=1}^{n}\nu_{a_j}\prod_{i<j}^{n}\zeta_{a_i a_j}(\beta_i - \beta_j) \tag{1.6}$$

$$\begin{aligned} F^{\phi_{1,3}}_{a_1\ldots a_n}(\beta_1, \ldots, \beta_n) &= \frac{2\cos(\pi/h)}{m_1}\left(\sum_{j=1}^{n} m_{a_j}\mathrm{e}^{\pm\beta_j}\right) f_{\pm;a_1\ldots a_n}(\beta_1, \ldots, \beta_n) \\ &\quad\times\prod_{j=1}^{n}\nu_{a_j}\prod_{i<j}^{n}\zeta_{a_i a_j}(\beta_i - \beta_j). \end{aligned} \tag{1.7}$$

The explicit forms of the constants ν_a, the functions $f_{\lambda;a_1\ldots a_n}(\beta_1, \ldots, \beta_n)$ ($\lambda = 0, \pm 1$) and $\zeta_{ab}(\beta)$ are given in section 2.1. Note that (1.7) gives two equivalent definitions of $\phi_{1,3}$.

For other operators, the explicit form of multiparticle form factors were determined only when $a_1 = \cdots = a_n = 1$ [19]. The explicit form for the form factors containing the other particle species had not been known. These form factors used to be given indirectly by using the fusion procedure. We will derive these in this paper.

After Wick rotation to the Euclidean space, the two-point correlation function of the operator $\phi_{1,1+s}$ and $\phi_{1,1+s'}$ can be represented as an infinite series of form factor contributions

$$\begin{aligned} \langle\phi_{1,1+s}(x)\phi_{1,1+s'}(0)\rangle &= \sum_{n=0}^{\infty}\sum_{a_i=1}^{N}\int\frac{\mathrm{d}^n\beta}{n!(2\pi)^n}\langle\mathrm{vac}\,|\phi_{1,1+s}(x)|\beta_1, \ldots, \beta_n\rangle_{a_1\ldots a_n}\ {}^{a_n\ldots a_1} \\ &\quad\times\langle\beta_n, \ldots, \beta_1|\phi_{1,1+s'}(0)|\,\mathrm{vac}\rangle \\ &= \sum_{n=0}^{\infty}\sum_{a_i}\int\frac{\mathrm{d}^n\beta}{n!(2\pi)^n}F^{\phi_{1,1+s}}_{a_1\ldots a_n}(\beta_1, \ldots, \beta_n)F^{\phi_{1,1+s'}}_{a_n\ldots a_1}(\beta_n + \pi\mathrm{i}, \ldots, \beta_1 + \pi\mathrm{i}) \\ &\quad\times\exp\left[-r\sum_{j=1}^{n} m_{a_j}\cosh\beta_j\right] \end{aligned} \tag{1.8}$$

where $r = (x^\mu x_\mu)^{1/2}$. In the next section, we transform (1.6) and (1.7) to forms which are convenient to sum up the series (1.8). From the final expression, we can guess the form of the form factors for the other off-critical primary operators. We give the conjectured form factors for $\phi_{1,1+s}$ and demonstrate that they satisfy form factor bootstrap equations. We discuss the relation between the conjectured form factors and their known forms with all $a_i = 1$ given by Koubek [19].

This paper is organized as follows. In the first part of section 2, a brief review of the form factor bootstrap equations is given. In section 2.1, the form factors (1.6) and (1.7) are transformed to a form which is convenient for summation. In section 2.2, we give the form factors for other primary operators $\phi_{1,1+s}$. In section 3, with the help of dual fields which act on an auxiliary Fock space, we sum up the infinite series (1.8) to a Fredholm determinant. Section 4 is devoted to discussion. In the appendix we give the evidence that the proposed form factors of $\phi_{1,1+s}$ satisfy the form factor bootstrap equations.

2. Form factor

To fix a notation, we briefly summarize the form factor bootstrap equations [12, 13].

The form factor bootstrap equations are axiomized in the following way.

(i) Watson's equations:

$$\begin{aligned} F^{\mathcal{O}}_{a_1\ldots a_i a_{i+1}\ldots a_n}&(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) \\ &= S_{a_i a_{i+1}}(\beta_i-\beta_{i+1}) F^{\mathcal{O}}_{a_1\ldots a_{i+1} a_i\ldots a_n}(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n) \end{aligned} \tag{2.1}$$

$$F^{\mathcal{O}}_{a_1 a_2\ldots a_n}(\beta_1+2\pi\mathrm{i},\beta_2,\ldots,\beta_n) = F^{\mathcal{O}}_{a_2\ldots a_n a_1}(\beta_2,\ldots,\beta_n,\beta_1). \tag{2.2}$$

(ii) Lorentz covariance:

$$F^{\mathcal{O}}_{a_1\ldots a_n}(\beta_1+\Lambda,\ldots,\beta_n+\Lambda) = \mathrm{e}^{s(\mathcal{O})\Lambda} F^{\mathcal{O}}_{a_1\ldots a_n}(\beta_1,\ldots,\beta_n) \tag{2.3}$$

where $s(\mathcal{O})$ is the Lorentz spin of the operator $\mathcal{O}$. The off-critical primary fields are scalar operators : $s(\phi_{1,1+s}) = 0$.

(iii) The kinematical residue equation:

$$-\mathrm{i}\lim_{\epsilon\to 0}\epsilon F^{\mathcal{O}}_{aad_1\ldots d_n}(\beta+\pi\mathrm{i}+\epsilon,\beta,\beta_1,\ldots,\beta_n) = \left(1-\prod_{j=1}^{n} S_{ad_j}(\beta-\beta_j)\right) F^{\mathcal{O}}_{d_1\ldots d_n}(\beta_1,\ldots,\beta_n). \tag{2.4}$$

(iv) Bound state residue equation: for a fusion process $a\times b\to c$, form factors satisfy the bound state residue equation

$$-\mathrm{i}\lim_{\epsilon\to 0}\epsilon F^{\mathcal{O}}_{abd_1\ldots a_n}(\beta+\mathrm{i}\bar{\theta}^{b}_{ac}+\epsilon,\beta-\mathrm{i}\bar{\theta}^{a}_{bc},\beta_1,\ldots,\beta_n) = \Gamma^{c}_{ab} F^{\mathcal{O}}_{cd_1\ldots d_n}(\beta,\beta_1,\ldots,\beta_n) \tag{2.5}$$

where $\bar{\theta} = \pi-\theta$ and θ^{c}_{ab} is the fusion angle. Let $n(a,b) = \min(a+b, h-a-b)$. In the perturbed minimal models, the fusion process occurs for $c = n(a,b)$ or $c = |a-b|(\neq 0)$ and the fusion angles are [33]

$$\begin{aligned} \theta^{|a-b|}_{ab} &= (h-|a-b|)\pi/h \\ \theta^{n(a,b)}_{ab} &= (a+b)\pi/h. \end{aligned} \tag{2.6}$$

The on-shell three-point coupling constant Γ^{c}_{ab} is given by

$$S_{ab}(\beta) \sim \frac{\mathrm{i}(\Gamma^{c}_{ab})^2}{\beta-\mathrm{i}\theta^{c}_{ab}} \qquad \text{for } \beta\sim\mathrm{i}\theta^{c}_{ab}. \tag{2.7}$$

Because the perturbed minimal model is non-unitary, the three-point coupling constant is pure imaginary for the case $c = h-a-b$ $(a+b > N)$ [33].

The S-matrix (1.3) has a double pole at $\beta = (a+b-2c)\pi\mathrm{i}/h$ for $c = 1,\ldots,\min(a,b)-1$ which corresponds to a weak bound state $a\times b\to((a-c)\times c)\times((b-c)\times c)$ [33].

Corresponding to this double pole, the form factor has a simple pole at certain rapidity difference. We do not give the explicit form of the (weak) bound state residue equations, which can be found in [13, 14].

As was shown by Koubek [19], it is sufficient to consider the minimal fusion process $a \times b \to a + b$ $(a + b < N)$. Information about the other fusion processes are indirectly contained in the minimal ones.

The explicit form of the minimal bound state residue equation is

$$
\begin{aligned}
&-\mathrm{i} \lim_{\epsilon \to 0} \epsilon F^{\mathcal{O}}_{abd_1 \ldots d_n}(\beta + b\pi\mathrm{i}/h + \epsilon, \beta - a\pi\mathrm{i}/h, \beta_1, \ldots, \beta_n) \\
&\qquad = \Gamma^{(a+b)}_{ab} F^{\mathcal{O}}_{(a+b)d_1 \ldots d_n}(\beta, \beta_1, \ldots, \beta_n) \qquad a + b \leqslant N
\end{aligned}
\tag{2.8}
$$

where

$$
(\Gamma^{(a+b)}_{ab})^2 = 2\tan((a+b)\pi/h) \frac{\tan(\max(a,b)\pi/h)}{\tan(\min(a,b)\pi/h)} \prod_{k=1}^{\min(a,b)-1} \left(\frac{\tan((\max(a,b)+k)\pi/h)}{\tan((\min(a,b)-k)\pi/h)} \right)^2 .
$$

The rest of the bound state residue equations can be derived from (2.8).

(v) Cluster properties [14, 17, 18, 34]:

$$
\begin{aligned}
&\lim_{\Lambda \to \infty} F^{\phi_{1,1+s}}_{a_1 \ldots a_m a_{m+1} \ldots a_{m+n}}(\beta_1 + \Lambda, \ldots, \beta_m + \Lambda, \beta_{m+1}, \ldots, \beta_{m+n}) \\
&\qquad = \frac{1}{\langle \phi_{1,1+s} \rangle} F^{\phi_{1,1+s}}_{a_1 \ldots a_m}(\beta_1, \ldots, \beta_m) F^{\phi_{1,1+s}}_{a_{m+1} \ldots a_{m+n}}(\beta_{m+1}, \ldots, \beta_{m+n}).
\end{aligned}
\tag{2.9}
$$

Here $\langle \phi_{1,1+s} \rangle$ is the vacuum expectation value of the off-critical primary operator $\phi_{1,1+s}$ [34]. We choose the normalization as follows:

$$
\langle \phi_{1,1+s} \rangle = 1. \tag{2.10}
$$

2.1. Form factors for $\phi_{1,2}$ and $\phi_{1,3}$

As was mentioned in the previous section, the form factors for $\phi_{1,2}$ and $\phi_{1,3}$ are given in the form (1.6) and (1.7) respectively.

The auxiliary objects $f_{\lambda;a_1 \ldots a_n}(\beta_1, \ldots, \beta_n)$ are defined by

$$
\begin{aligned}
f_{\lambda;a_1 \ldots a_n}(\beta_1, \ldots, \beta_n) &= (-1)^{n-1} 2 \int_{\Gamma_{a_1}(\beta_1)} \frac{\mathrm{d}\alpha_1}{2\pi\mathrm{i}} \cdots \int_{\Gamma_{a_{n-1}}(\beta_{n-1})} \frac{\mathrm{d}\alpha_{n-1}}{2\pi\mathrm{i}} \\
&\quad \times \prod_{i=1}^{n-1} \prod_{j=1}^{n} \varphi_{a_j}(\alpha_i - \beta_j) \prod_{i<j}^{n-1} \sinh(\alpha_i - \alpha_j) \\
&\quad \times \exp\left(\lambda (\sum_{i=1}^{n-1} \alpha_i - \sum_{j=1}^{n} \beta_j) \right) \qquad \lambda = 0, \pm 1
\end{aligned}
\tag{2.11}
$$

where $\Gamma_a(\beta)$ is the contour enveloping the points $\beta + (a - 2l)\pi\mathrm{i}/h$, $l = 0, 1, \ldots, a$ and

$$
\varphi_a(\beta) = \frac{\prod_{j=1}^{a-1} \cosh \frac{1}{2}(\beta + (a-2j)\pi\mathrm{i}/h)}{2 \prod_{j=0}^{a} \sinh \frac{1}{2}(\beta + (a-2j)\pi\mathrm{i}/h)}. \tag{2.12}
$$

Note that our $f_{\lambda;a_1 \ldots a_n}(\beta_1, \ldots, \beta_n)$ corresponds to Smirnov's $F_{-\lambda}(\beta_1, \ldots, \beta_n)_{a_1 \ldots a_n}$ [14]. Although the integration contour $\Gamma_{a_n}(\beta_n)$ is absent in the expression (2.11), all rapidities are on the same footing in $f_{\lambda;a_1 \ldots a_n}$. See (2.36).

The function $\zeta_{ab}(\beta)$ is defined by

$$
\zeta_{ab}(\beta) = W_{ab}(\beta) F^{(\min)}_{ab}(\beta) \tag{2.13}
$$

where

$$W_{ab}(\beta) = (-1)^{a+\min(a,b)+1} \frac{2\prod_{j=0}^{|a-b|} \sinh \frac{1}{2}(\beta + (|a-b| - 2j)\pi \mathrm{i}/h)}{\prod_{j=1}^{a+b-1} \cosh \frac{1}{2}(\beta + (a+b-2j)\pi \mathrm{i}/h)}. \tag{2.14}$$

The phase of $W_{ab}(\beta)$ is chosen such that the cluster equation (2.9) holds. The minimal two-body form factor $F_{ab}^{(\min)}(\beta)$ is given by

$$F_{ab}^{(\min)}(\beta) = \prod_{\substack{x=|a-b|+1 \\ \text{step}2}}^{a+b-1} F_x^{(\min)}(\beta). \tag{2.15}$$

Here $F_x^{(\min)}(\beta)$ is a building block of the minimal two-body form factor:

$$F_x^{(\min)}(\beta) = N_x \exp\left(4\int_0^\infty \frac{\mathrm{d}k}{k} \frac{\sin^2(\hat{\beta}k/2\pi)\cosh(1/2 - x/h)k \cosh(k/h)}{\cosh(k/2)\sinh k}\right) \tag{2.16}$$

where $\hat{\beta} = \pi\mathrm{i} - \beta$ and a normalization constant N_x is chosen as

$$N_x = \exp\left(2\int_0^\infty \frac{\mathrm{d}k}{k} \frac{\cosh(k/2) - \cosh(1/2 - x/h)k\cosh(k/h)}{\cosh(k/2)\sinh k}\right). \tag{2.17}$$

$F_x^{(\min)}(\beta)$ has no poles or no zeros in the strip $0 < \mathrm{Im}\,\beta < 2\pi$. $F_1^{(\min)}(\beta)$ has a single zero at $\beta = 0$.

The constant ν_a is defined by

$$\nu_a = i^a \left(\frac{2\sin(2a\pi/h)}{\pi F_{aa}^{(\min)}(\pi i)}\right)^{1/2} \prod_{l=1}^{a-1} \sin(l\pi/h). \tag{2.18}$$

Then the functions (1.6) and (1.7) with (2.11), (2.13) and (2.18) satisfy the form factor bootstrap equations and are indeed form factors for $\phi_{1,2}$ and $\phi_{1,3}$ respectively [14].

In order to transform the form factors into forms suited for summation, we rewrite $f_{\lambda;a_1\ldots a_n}(\beta_1, \ldots, \beta_n)$ in terms of $t_i = \mathrm{e}^{\alpha_i}$ and $x_j = \mathrm{e}^{\beta_j}$. Let $\omega = \exp(2\pi\mathrm{i}/h)$. We have

$$\begin{aligned} f_{\lambda;a_1\ldots a_n}(\beta_1, \ldots, \beta_n) &= (-1)^{n-1} 2^{n(n+1)/2} \prod_{j=1}^{n} x_j^{n-\lambda-1} \int_{\gamma_{a_1}(x_1)} \frac{\mathrm{d}t_1}{2\pi\mathrm{i}} \\ &\quad \cdots \int_{\gamma_{a_{n-1}}(x_{n-1})} \frac{\mathrm{d}t_{n-1}}{2\pi\mathrm{i}} \prod_{i=1}^{n-1}\prod_{j=1}^{n} \varphi_{a_j}(t_i, x_j) \prod_{i<j}^{n-1}(t_i^2 - t_j^2) \prod_{i=1}^{n-1} t_i^{\lambda+1} \end{aligned} \tag{2.19}$$

where the contour $\gamma_a(x)$ envelops the points $x\omega^{a/2-l}$ for $l = 0, \ldots, a$. For $t = \mathrm{e}^{\alpha}$ and $x = \mathrm{e}^{\beta}$, the function $\varphi_a(t, x)$ is defined by $\varphi_a(\alpha - \beta) = 2tx\varphi_a(t, x)$. The explicit form of $\varphi_a(t, x)$ is given by

$$\varphi_a(t, x) = \frac{\prod_{j=1}^{a-1}(t + x\omega^{a/2-j})}{\prod_{j=0}^{a}(t - x\omega^{a/2-j})}. \tag{2.20}$$

With help of a Vandermonde determinant

$$\prod_{i>j}^{n-1}(t_i^2 - t_j^2) = \det(t_i^{2j-2})_{1\leqslant i,j\leqslant n-1} \tag{2.21}$$

we can write (2.19) in the following form:

$$f_{\lambda;a_1\ldots a_n}(\beta_1, \ldots, \beta_n) = (-1)^{n(n-1)/2} 2^{n(n+1)/2} \prod_{j=1}^{n} x_j^{n-\lambda-1} \det(K^{\lambda}_{a_1\ldots a_n;ij})_{1\leqslant i,j\leqslant n-1} \tag{2.22}$$

where

$$K^{\lambda}_{a_1\ldots a_n;ij} = \int_{\gamma_{a_i}(x_i)} \frac{\mathrm{d}t}{2\pi\mathrm{i}} t^{2j+\lambda-1} \prod_{k=1}^{n} \varphi_{a_k}(t, x_k) \qquad i, j = 1, \ldots, n-1. \tag{2.23}$$

The contour $\gamma_{a_i}(x_i)$ envelops the points $t = x_i \omega^{a_i/2-l}$ for $l = 0, \ldots, a_i$.

Following the procedure of [14], we transform the determinant of $K^{\lambda}_{a_1\ldots a_n}$ (2.23). Let us consider the properties of (2.23). The pole structure of the integrand is determined by

$$\prod_{k=1}^{n} \varphi_{a_k}(t, x_k). \tag{2.24}$$

The function $\varphi_{a_k}(t, x_k)$ $(k \neq i)$ has no pole in the contour $\gamma_{a_i}(x_i)$. Thus the value of the integral does not change if $\varphi_{a_k}(t, x_k)$ $(k \neq i)$ is replaced by

$$\frac{t^h - (-1)^{a_k} x_k^h}{(-1)^{a_i} x_i^h - (-1)^{a_k} x_k^h} \varphi_{a_k}(t, x_k). \tag{2.25}$$

Then we have

$$\begin{aligned} K^{\lambda}_{a_1\ldots a_n;ij} &= \int_{\gamma_{a_i}(x_i)} \frac{\mathrm{d}t}{2\pi\mathrm{i}} t^{2j+\lambda-1} \varphi_{a_i}(t, x_i) \prod_{k\neq i}^{n} \frac{(t^h - (-1)^{a_k} x_k^h)\varphi_{a_k}(t, x_k)}{(-1)^{a_i} x_i^h - (-1)^{a_k} x_k^h} \\ &= \prod_{k\neq i}^{n} \frac{1}{(-1)^{a_i} x_i^h - (-1)^{a_k} x_k^h} \int_{\gamma_{a_i}(x_i)} \frac{\mathrm{d}t}{2\pi\mathrm{i}} t^{2j+\lambda-1} \prod_{k=1}^{n} \psi_{a_k}(t, x_k) \frac{1}{t^h - (-1)^{a_i} x_i^h} \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} \psi_a(t, x) &= (t^h - (-1)^a x^h)\varphi_a(t, x) \\ &= \prod_{j=1}^{a-1} (t + x\omega^{a/2-j}) \prod_{j=1}^{h-a-1} (t + x\omega^{(h-a)/2-j}). \end{aligned} \tag{2.27}$$

Now the integrand in the integral over t is regular at the point $t = 0$ and has no singularities except for the points $x_i \omega^{a_i/2-l}$ for $l = 0, \ldots, a_i$. Thus we can replace the contour $\gamma_{a_i}(x_i)$ by a circle whose radius is larger than $|x_i|$. Then we have

$$\det(K^{\lambda}_{a_1\ldots a_n;ij})_{1\leqslant i,j\leqslant n-1} = \prod_{i=1}^{n-1} \prod_{j=1(\neq i)}^{n} ((-1)^{a_i} x_i^h - (-1)^{a_j} x_j^h)^{-1} \det(\tilde{K}^{\lambda}_{a_1\ldots a_n;ij})_{1\leqslant i,j\leqslant n-1} \tag{2.28}$$

where

$$\tilde{K}^{\lambda}_{a_1\ldots a_n;ij} = \oint \frac{\mathrm{d}t}{2\pi\mathrm{i}} t^{2j+\lambda-1} \prod_{k=1}^{n} \psi_{a_k}(t, x_k) \frac{1}{t^h - (-1)^{a_i} x_i^h}. \tag{2.29}$$

On the contour it holds that $|t| > |x_j|$. So we can expand $(t^h - (-1)^{a_i} x_i^h)^{-1}$ as follows:

$$\frac{1}{t^h - (-1)^{a_i} x_i^h} = \sum_{q=1}^{\infty} (-1)^{a_i(q-1)} x_i^{h(q-1)} t^{-hq}. \tag{2.30}$$

Note that after substitution of the above equation into the integral (2.29), terms with $q \geqslant n$ vanish because the highest degree of the integrand in t is smaller than -1. The number of non-vanishing terms is at most $n - 1$:

$$\tilde{K}^{\lambda}_{a_1\ldots a_n;ij} = \sum_{q=1}^{n-1} (-1)^{a_i(q-1)} x_i^{h(q-1)} \oint \frac{\mathrm{d}t}{2\pi\mathrm{i}} t^{2j+\lambda-1-hq} \prod_{k=1}^{n} \psi_{a_k}(t, x_k). \tag{2.31}$$

The sum over q in the above equation can be interpreted as the matrix product of two matrices of dimension $n-1$. The determinant of $\tilde{K}^{\lambda}_{a_1\ldots a_n}$ becomes a product of two determinants:

$$\begin{aligned}\det(\tilde{K}^{\lambda}_{a_1\ldots a_n;ij})_{1\leqslant i,j\leqslant n-1} &= \det((-1)^{a_i(q-1)}x_i^{h(q-1)})_{1\leqslant i,q\leqslant n-1}\\ &\quad\times\det\left(\oint\frac{\mathrm{d}t}{2\pi\mathrm{i}}t^{2j+\lambda-1-hq}\prod_{k=1}^{n}\psi_{a_k}(t,x_k)\right)_{1\leqslant q,j\leqslant n-1}\\ &= \prod_{i>j}^{n-1}((-1)^{a_i}x_i^h-(-1)^{a_j}x_j^h)\det\left(\oint\frac{\mathrm{d}t}{2\pi\mathrm{i}}t^{2j+\lambda-1-hi}\prod_{k=1}^{n}\psi_{a_k}(t,x_k)\right)_{1\leqslant i,j\leqslant n-1}.\end{aligned} \tag{2.32}$$

To make the meaning of the determinant in (2.32) clear, it is useful to introduce a notion of 'generalized' elementary symmetric polynomials. Recall that the elementary symmetric polynomials with m variables are defined by

$$\prod_{k=1}^{m}(t+z_k)=\sum_{k\in\mathbf{Z}}t^{m-k}\sigma_k^{(m)}(z_1,\ldots,z_m). \tag{2.33}$$

It holds that $\sigma_k^{(m)}=0$ if $k<0, k>m$. Similarly, let us define generalized elementary symmetric polynomials by

$$\prod_{k=1}^{n}\psi_{a_k}(t,x_k)=\sum_{k\in\mathbb{Z}}t^{(h-2)n-k}E_{a_1\ldots a_n;k}(x_1,\ldots,x_n). \tag{2.34}$$

Using the definition of $\psi_a(t,x)$ (2.27), we can express the generalized elementary symmetric polynomial in terms of the ordinary elementary symmetric polynomials with $(h-2)n$ variables:

$$\begin{aligned}E_{a_1\ldots a_n;k}(x_1,\ldots,x_n)=\sigma_k^{((h-2)n)}\Big(&\overbrace{\omega^{a_1/2-1}x_1,\omega^{a_1/2-2}x_1,\ldots,\omega^{-a_1/2+1}x_1,}^{a_1-1}\\ &\overbrace{\omega^{(h-a_1)/2-1}x_1,\omega^{(h-a_1)/2-2}x_1,\ldots,\omega^{-(h-a_1)/2+1}x_1,}^{h-a_1-1}\\ &\ldots,\\ &\overbrace{\omega^{a_n/2-1}x_n,\omega^{a_n/2-2}x_n,\ldots,\omega^{-a_n/2+1}x_n,}^{a_n-1}\\ &\overbrace{\omega^{(h-a_n)/2-1}x_n,\omega^{(h-a_n)/2-2}x_n,\ldots,\omega^{-(h-a_n)/2+1}x_n}^{h-a_n-1}\Big).\end{aligned}$$

Note that $E_{a_1\ldots a_n;k}=0$ for $k<0$ or $k>(h-2)n$.

For $N=1$, the generalized elementary symmetric polynomials coincide with the ordinary symmetric polynomials.

The determinant can be written as follows:

$$\begin{aligned}\det\left(\oint\frac{\mathrm{d}t}{2\pi\mathrm{i}}t^{2j+\lambda-1-hi}\prod_{k=1}^{n}\psi_{a_k}(t,x_k)\right)_{1\leqslant i,j\leqslant n-1}&=\det(E_{a_1\ldots a_n;h(n-i)-2(n-j)+\lambda})_{1\leqslant i,j\leqslant n-1}\\ &=\det(E_{a_1\ldots a_n;hi-2j+\lambda})_{1\leqslant i,j\leqslant n-1}.\end{aligned} \tag{2.35}$$

Recall that the form factor in the scaling Lee–Yang model ($N = 1$, $h = 3$) was proportional to $\det(\sigma^{(n)}_{3i-2j+\lambda})_{1\leqslant i,j\leqslant n-1}$ [14, 15]. Thus, the expression (2.35) is the natural generalization of the $N = 1$ case.

Then, we have a representation of $f_{\lambda;a_1\ldots a_n}$:

$$f_{\lambda;a_1\ldots a_n}(\beta_1,\ldots,\beta_n) = 2^{n(n+1)/2}\prod_{j=1}^{n} x_j^{n-\lambda-1}\prod_{i>j}^{n}((-1)^{a_i}x_i^h-(-1)^{a_j}x_j^h)^{-1} \times\det(E_{a_1\ldots a_n;hi-2j+\lambda})_{1\leqslant i,j\leqslant n-1} \qquad \lambda = 0, \pm 1. \tag{2.36}$$

As was shown in [26, 27], in order to represent two-point correlation function as a Fredholm determinant, it is necessary to transform the determinant of the matrix of dimension $n-1$ into a determinant of a matrix of dimension n.

Let us consider the following matrix:

$$M^{\lambda}_{a_1\ldots a_n;ij}(x_1,\ldots,x_n) = E_{a_1\ldots a_n;hi-2j+\lambda-h+1}(x_1,\ldots,x_n) \qquad i, j = 1,\ldots,n. \tag{2.37}$$

For $\lambda = 0$ or 1, it holds that

$$M^{\lambda+1}_{a_1\ldots a_n;1j} = E_{a_1\ldots a_n;\lambda}\delta_{j,1} \qquad j = 1,\ldots,n \qquad \lambda = 0, 1 \tag{2.38}$$

and $M^{\lambda+1}_{a_1\ldots a_n;(i+1)(j+1)} = E_{a_1\ldots a_n;hi-2j+\lambda}$ for $1 \leqslant i, j \leqslant n-1$. Thus we have

$$\det(M^{\lambda+1}_{a_1\ldots a_n;ij})_{1\leqslant i,j\leqslant n} = E_{a_1\ldots a_n;\lambda}\det(E_{a_1\ldots a_n;hi-2j+\lambda})_{1\leqslant i,j\leqslant n-1} \qquad \lambda = 0, 1. \tag{2.39}$$

For $\lambda = -1$, it holds that

$$M^{\lambda+h-1}_{a_1\ldots a_n;nj} = E_{a_1\ldots a_n;(h-2)n-1}\delta_{n,j} \qquad j = 1,\ldots,n, \qquad \lambda = -1 \tag{2.40}$$

and $M^{\lambda+h-1}_{a_1\ldots a_n;ij} = E_{a_1\ldots a_n;hi-2j+\lambda}$ for $1 \leqslant i, j \leqslant n-1$. Thus we have

$$\det(M^{\lambda+h-1}_{a_1\ldots a_n;ij})_{1\leqslant i,j\leqslant n} = E_{a_1\ldots a_n;(h-2)n-1}\det(E_{a_1\ldots a_n;hi-2j+\lambda})_{1\leqslant i,j\leqslant n-1} \qquad \lambda = -1. \tag{2.41}$$

Note that

$$E_{a_1\ldots a_n;0} = 1 \tag{2.42}$$

$$E_{a_1\ldots a_n;1} = 2\cos(\pi/h)\sum_{j=1}^{n}\frac{\sin(a_j\pi/h)}{\sin(\pi/h)}x_j = \frac{2\cos(\pi/h)}{m_1}\left(\sum_{j=1}^{n} m_{a_j}\mathrm{e}^{\beta_j}\right) \tag{2.43}$$

$$E_{a_1\ldots a_n;(h-2)n-1}(x_1,\ldots,x_n) = \left(\prod_{j=1}^{n} x_j^{h-2}\right)E_{a_1\ldots a_n;1}(x_1^{-1},\ldots,x_n^{-1}). \tag{2.44}$$

Combining the above results with (1.6) and (1.7), the form factors of $\phi_{1,1+s}$ ($s = 1, 2$) can be rewritten as

$$F^{\phi_{1,1+s}}_{a_1\ldots a_n}(\beta_1,\ldots,\beta_n) = \tilde{f}_{\lambda;a_1\ldots a_n}(\beta_1,\ldots,\beta_n)\left(\prod_{j=1}^{n}\nu_{a_j}\right)\prod_{i<j}^{n}\zeta_{a_i a_j}(\beta_i-\beta_j) \qquad s = 1, 2 \tag{2.45}$$

where $\lambda = 1$ for $s = 1$, and $\lambda = 2$ or $h-2$ for $s = 2$ and

$$\tilde{f}_{\lambda;a_1\ldots a_n}(\beta_1,\ldots,\beta_n) = 2^{n(n+1)/2}\prod_{j=1}^{n} x_j^{n-\lambda}\prod_{i>j}^{n}((-1)^{a_i}x_i^h-(-1)^{a_j}x_j^h)^{-1}\det(M^{\lambda}_{a_1\ldots a_n;ij})_{1\leqslant i,j\leqslant n}. \tag{2.46}$$

The above auxiliary object has an integral representation similar to f_λ (2.11):

$$\tilde{f}_{\lambda;a_1\ldots a_n}(\beta_1,\ldots,\beta_n)=\int_{\Gamma_{a_1}(\beta_1)}\frac{\mathrm{d}\alpha_1}{2\pi\mathrm{i}}\cdots\int_{\Gamma_{a_n}(\beta_n)}\frac{\mathrm{d}\alpha_n}{2\pi\mathrm{i}}\prod_{i=1}^{n}\prod_{j=1}^{n}\varphi_{a_j}(\alpha_i-\beta_j)\times\prod_{i<j}^{n}\sinh(\alpha_i-\alpha_j)\exp\left(\lambda\sum_{j=1}^{n}(\alpha_j-\beta_j)\right). \qquad (2.47)$$

In contrast to (2.11), this expression treats all β_i on equal footing. The equivalence of (2.47) to (2.46) can be proven in exactly the same way as for the case of $f_{\lambda;a_1\ldots a_n}$.

2.2. Form factors for $\phi_{1,\,1+s}$

Let us analyse properties of (2.46) more closely.

Except for $\lambda=1,\ldots,2N$, $\det M^{\lambda}_{a_1\ldots a_n}$ are trivial: $\det M^{\lambda}_{a_1\ldots a_n}=\delta_{n,0}$.

From the definition of the generalized elementary symmetric polynomials (2.34), we can show that

$$E_{a_1\ldots a_n;k}(x_1,\ldots,x_n)=\left(\prod_{j=1}^{n}x_j^{h-2}\right)E_{a_1\ldots a_n;(h-2)n-k}(x_1^{-1},\ldots,x_n^{-1}). \qquad (2.48)$$

The matrix $M^{h-\lambda}_{a_1\ldots a_n}$ is 'isomorphic' to the matrix $M^{\lambda}_{a_1\ldots a_n}$ in the sense

$$M^{h-\lambda}_{a_1\ldots a_n;ij}(x_1,\ldots,x_n)=\left(\prod_{j=1}^{n}x_j^{h-2}\right)M^{\lambda}_{a_1\ldots a_n;(n+1-i)(n+1-j)}(x_1^{-1},\ldots,x_n^{-1}). \qquad (2.49)$$

Further, it holds that

$$\left(\prod_{j=1}^{n}x_j^{-\lambda}\right)\det(M^{\lambda}_{a_1\ldots a_n}(x_1,\ldots,x_n))=\left(\prod_{j=1}^{n}x_j^{\lambda-h}\right)\det(M^{h-\lambda}_{a_1\ldots a_n}(x_1,\ldots,x_n)). \qquad (2.50)$$

Thus, we have

$$\tilde{f}_{h-\lambda;a_1\ldots a_n}(\beta_1,\ldots,\beta_n)=\tilde{f}_{\lambda;a_1\ldots a_n}(\beta_1,\ldots,\beta_n). \qquad (2.51)$$

It is now easy to guess the form of the form factors for the general off-critical primary fields $\phi_{1,1+s}$ ($s=0,\ldots,2N+1$). Suppose that the form factors of $\phi_{1,1+s}$ are given by

$$F^{\phi_{1,1+s}}_{a_1\ldots a_n}(\beta_1,\ldots,\beta_n)=\tilde{f}_{s;a_1\ldots a_n}(\beta_1,\ldots,\beta_n)\left(\prod_{j=1}^{n}\nu_{a_j}\right)\prod_{i<j}^{n}\zeta_{a_ia_j}(\beta_i-\beta_j) \qquad (2.52)$$

where $\tilde{f}_{s;a_1\ldots a_n}$ is given by equation (2.46)

$$\tilde{f}_{s;a_1\ldots a_n}(\beta_1,\ldots,\beta_n)=2^{n(n+1)/2}\prod_{j=1}^{n}x_j^{n-s}\prod_{i>j}^{n}((-1)^{a_i}x_i^h-(-1)^{a_j}x_j^h)^{-1}\det(M^{s}_{a_1\ldots a_n;ij})_{1\leqslant i,j\leqslant n} \qquad (2.53)$$

and $M^{s}_{a_1\ldots a_n;ij}$ is given by equation (2.37)

$$M^{s}_{a_1\ldots a_n;ij}(x_1,\ldots,x_n)=E_{a_1\ldots a_n;hi-2j+s-h+1}(x_1,\ldots,x_n)\qquad i,j=1,\ldots n. \qquad (2.54)$$

Recall that the definitions of the constant ν_a and the function $\zeta_{ab}(\beta)$ are given by equation (2.18) and equation (2.13) respectively.

In the appendix, we demonstrate that (2.52) satisfies the form factor bootstrap equations.

The form of the form factor bootstrap equations does not depend on the operator. We need to identify the solution with some operator. The justification of the operator identification in (2.52) is the following: From equation (2.51), it holds that $\phi_{1,1+s} = \phi_{1,1+h-s}$. For $s = 0$ or $s = 2N+1$ case, the off-critical primary field is the identity operator and the above form factors give trivial solution. For $s = 1, 2, 2N-1$, equation (2.52) yields the known results [14]. For general s, let us consider the special case of (2.52):

$$F_n^{\phi_{1,1+s}}(\beta_1, \dots, \beta_n) := F_{1\dots1}^{\phi_{1,1+s}}(\beta_1, \dots, \beta_n). \tag{2.55}$$

The explicit form of the form factors of $\phi_{1,1+2k}$ for $a_1 = \cdots = a_n = 1$ can be found in [19]. We conjecture that (2.55) has another equivalent expression:

$$F_n^{\phi_{1,1+s}}(\beta_1, \dots, \beta_n) = (2\nu_1)^n [s]_{\omega^{1/2}} \det([s + 2i - 2j]_{\omega^{1/2}} \sigma_{2i-j}^{(n)})_{1\leqslant i,j\leqslant n-1}$$
$$\times \prod_{i<j}^{n} \frac{F_{11}^{(\min)}(\beta_i - \beta_j)}{(x_i + x_j)\sinh\frac{1}{2}(\beta_i - \beta_j + 2\pi\mathrm{i}/h)\sinh\frac{1}{2}(\beta_i - \beta_j - 2\pi\mathrm{i}/h)} \tag{2.56}$$

where

$$[n]_{\omega^{1/2}} = \frac{\omega^{n/2} - \omega^{-n/2}}{\omega^{1/2} - \omega^{-1/2}} = \frac{\sin(n\pi/h)}{\sin(\pi/h)}. \tag{2.57}$$

We checked that both (2.55) and (2.56) satisfy the same kinematical residue equations and give the same results for small n. If we set $s = 2k$, (2.56) agrees with the Koubek's results [19]. The scaling dimensions of the operators were checked numerically for small N in [35]. These results completely agree with our operator identification.

Thus the function (2.52) gives the form factor for $\phi_{1,1+s}$. Equation (2.52) is one of the main results of this paper.

For later convenience, we further rewrite the form factor (2.52) as follows:

$$F_{a_1\dots a_n}^{\phi_{1,1+s}}(\beta_1, \dots, \beta_n) = 2^n \left(\prod_{j=1}^{n} \nu_{a_j} x_j^{1-s}\right) \det(M_{a_1\dots a_n}^s) \prod_{i<j}^{n} \frac{\tilde{\zeta}_{a_i a_j}(\beta_i - \beta_j)}{(x_i x_j)^{(h-2)/2}} \tag{2.58}$$

where

$$\tilde{\zeta}_{ab}(\beta) = \tilde{W}_{ab}(\beta) F_{ab}^{(\min)}(\beta) \tag{2.59}$$

$$\frac{2W_{ab}(\beta - \beta')}{(-1)^b y^h - (-1)^a x^h} = (xy)^{-h/2} \tilde{W}_{ab}(\beta - \beta') \qquad x = \mathrm{e}^{\beta},\ y = \mathrm{e}^{\beta'}. \tag{2.60}$$

The explicit form of $\tilde{W}_{ab}$ is given by

$$\tilde{W}_{ab}(\beta) = \frac{\mathrm{i}^{|a-b|}}{\sinh\frac{1}{2}h(\beta + (a+b)\pi\mathrm{i}/h)} \frac{2\prod_{j=0}^{|a-b|}\sinh\frac{1}{2}(\beta + (|a-b| - 2j)\pi\mathrm{i}/h)}{\prod_{j=1}^{a+b-1}\cosh\frac{1}{2}(\beta + (a+b-2j)\pi\mathrm{i}/h)}. \tag{2.61}$$

In the next section, using the expression (2.58), we sum up the two-point correlation function (1.8) into a Fredholm determinant of an integral operator.

3. The determinant representation

We can write correlation functions using a Fredholm determinant with the help of auxiliary operators. We only give the final result which is straight generalization of the case of the sinh-Gordon model [26] and the scaling Lee–Yang model [27].

The two-point correlation functions can be written as

$$\langle\phi_{1,1+s}(x)\phi_{1,1+s'}(0)\rangle = (0|\det(I + \hat{U}^{(s;s')})|0) \tag{3.1}$$

where an integral operator is defined by

$$\hat{U}^{(s;s')}(y,z) = 2h\pi^{-1}(yz)^{(h-s-s'+1)/2} \oint \frac{\mathrm{d}t^2}{(2\pi\mathrm{i})^2} \frac{t_1^s t_2^{s'}}{(t_1t_2)^h - 1} \frac{\mathrm{e}^{\Phi(y)}}{t_1^2+y^2} \frac{\mathrm{e}^{\Phi(z)}}{t_2^2+z^2}. \tag{3.2}$$

The auxiliary quantum operators are defined as

$$\Phi(y) = \sum_{a=1}^{N} \Phi_{1a}(y) + \tfrac{1}{2}\Phi_0(y) \tag{3.3}$$

$$\mathrm{e}^{\Phi_0(y)} = \sum_{a=1}^{N} (-1)^a \exp[\Phi_{0a}(y) + \Phi_{2a}(y) - rm_a(y+y^{-1})/2] \tag{3.4}$$

with $r = (x^\mu x_\mu)^{1/2}$. Here $\Phi_{ja}(y)$ are mutually commuting operators given by the auxiliary operators: $\Phi_{0a}(y) = q_{0a}(y) + p_{0a}(y)$ and $\Phi_{ja}(y) = q_{ja}(y) + p_{(3-j)a}(y)$ $(j = 1, 2)$. The operators $p_{ja}(y)$ and $q_{ja}(y)$ act on the canonical Fock space in the following way

$$(0|q_{ja}(y) = 0 \qquad p_{ja}(y)|0) = 0 \qquad j = 0, 1, 2 \qquad a = 1, \ldots, N. \tag{3.5}$$

Non-zero commutators are given by

$$[p_{1a}(y), q_{1a}(z)] = [p_{2a}(y), q_{2a}(z)] = \log((y^2+z^2)\psi_a(y,z)) \tag{3.6}$$

$$[p_{0a}(y), q_{0b}(z)] = 2\log\left|\frac{\tilde{\zeta}_{ab}(\log(y/z))}{(yz)^{(h-2)/2}(y^2-z^2)}\right|. \tag{3.7}$$

4. Discussion

In this paper, we have considered the two-point correlation functions in the perturbed minimal models $M_{2,2N+3} + \phi_{1,3}$.

It is known that the operator content of the perturbed model is same as the unperturbed models [19]. The model contains $N+1$ off-critical primary fields $\phi_{1,1+s}$.

We have determined the explicit form of the form factors for the off-critical primary fields $\phi_{1,1+s}$ (2.52). The information about the operator $\phi_{1,1+s}$ is carried by the function $\tilde{f}_{s;a_1\ldots a_n}$ (2.47):

$$\tilde{f}_{s;a_1\ldots a_n}(\beta_1,\ldots,\beta_n) = \int_{\Gamma_{a_1}(\beta_1)} \frac{\mathrm{d}\alpha_1}{2\pi\mathrm{i}} \cdots \int_{\Gamma_{a_n}(\beta_n)} \frac{\mathrm{d}\alpha_n}{2\pi\mathrm{i}} \prod_{i=1}^{n}\prod_{j=1}^{n} \varphi_{a_j}(\alpha_i - \beta_j)$$
$$\times \prod_{i<j}^{n} \sinh(\alpha_i - \alpha_j) \exp\left(s\sum_{j=1}^{n}(\alpha_j - \beta_j)\right). \tag{4.1}$$

This representation reveals the remarkably simple structure of the operator content of the perturbed minimal model.

Recall that the perturbed minimal model can be described as the restriction of the sine-Gordon model at the coupling constant $g^2/8\pi = 2/(2N+3)$ [14]. In the restricted sine-Gordon model, the off-critical primary field $\phi_{1,1+s}$ corresponds to the following exponential operator:

$$\mathcal{P}\mathrm{e}^{\mathrm{i}sg\phi/2}\mathcal{P} \tag{4.2}$$

where ϕ is the sine-Gordon field and $\mathcal{P}$ is the projection operator into the soliton-free sector [14, 19]. If we use the representation (2.47) (not (2.46)) and replace $2\pi/h$ by $\xi = \pi g^2/(8\pi - g^2)$, the form factor (2.52) becomes the breather form factor for the exponential operator $\mathrm{e}^{\mathrm{i}sg\phi/2}$ in the unrestricted sine-Gordon model. The expression (2.56) remains valid by this replacement and it gives the form factors for the lightest breathers. It can be obtained from the form factors of the exponential operator in the sinh-Gordon model by analytic continuation in the coupling constant [24].

Using a representation of the form factor (2.58), we have obtained determinant representation for the two-point correlation function of off-critical primary fields (3.1), which is a natural generalization of that of the scaling Lee-Yang model [27].

It would be very interesting if one could extract some non-perturbative features from the determinant representations (3.1).

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Appendix

In this appendix, we collect some relations which are helpful to show that the function (2.52) satisfies the form factor bootstrap equations (i)–(v).

There is no difficulty in proving (i) Watson's equation and (ii) Lorentz covariance.

Note that the minimal building block of the two-body form factor $F_x^{(\min)}(\beta)$ (2.16) has a property

$$F_x^{(\min)}(\beta) = \{x\}_{(\beta)} F_x^{(\min)}(-\beta) \tag{A.1}$$

$$F_x^{(\min)}(\beta + 2\pi\mathrm{i}) = F_x^{(\min)}(-\beta). \tag{A.2}$$

Then the minimal two-body form factor $F_{ab}^{(\min)}(\beta)$ (2.15) satisfies Watson's equation for $n = 2$:

$$F_{ab}^{(\min)}(\beta) = S_{ab}(\beta) F_{ab}^{(\min)}(-\beta) \tag{A.3}$$

$$F_{ab}^{(\min)}(\beta + 2\pi\mathrm{i}) = F_{ab}^{(\min)}(-\beta). \tag{A.4}$$

Using these relations, one can easily check that (2.52) obeys Watson's equations for general n.

(iii) Kinematical residue equation: to show that (2.52) satisfies the kinematical residue equation, we need the following relations.

The residue of $\zeta_{aa}(\beta)$ at $\beta = \pi\mathrm{i}$ is given by

$$-\mathrm{i} \lim_{\epsilon\to 0} \epsilon \zeta_{aa}(\pi\mathrm{i} + \epsilon) = (-1)^{a-1} 4\mathrm{i} F_{aa}^{(\min)}(\pi\mathrm{i}) \prod_{j=1}^{a-1} \sin^{-2}(j\pi/h). \tag{A.5}$$

Using a representation of $\tilde{f}_\lambda$ (2.47), one can show that

$$\begin{aligned}\tilde{f}_{\lambda;aad_1\ldots d_n}(\beta + \pi\mathrm{i}, \beta, \beta_1, \ldots, \beta_n) = {} & \frac{\mathrm{i}}{\sin(2\pi a/h)} \\ & \times \Bigg[\prod_{j=1}^{n} \varphi_{d_j}(\beta - \beta_j + \pi\mathrm{i} - a\pi\mathrm{i}/h) \varphi_{d_j}(\beta - \beta_j + a\pi\mathrm{i}/h) \\ & - \prod_{j=1}^{n} \varphi_{d_j}(\beta - \beta_j + \pi\mathrm{i} + a\pi\mathrm{i}/h) \varphi_{d_j}(\beta - \beta_j - a\pi\mathrm{i}/h) \Bigg] \tilde{f}_{\lambda;d_1\ldots d_n}(\beta_1, \ldots, \beta_n).\end{aligned} \tag{A.6}$$

With the choice of the normalization (2.17), the building block (2.16) satisfies

$$F_x^{(\min)}(\beta+\pi\mathrm{i})F_x^{(\min)}(\beta) = -\cosh\tfrac{1}{2}(\beta-(x-1)\pi\mathrm{i}/h)\cosh\tfrac{1}{2}(\beta-(x+1)\pi\mathrm{i}/h)$$
$$\times\sinh\tfrac{1}{2}(\beta+(x-1)\pi\mathrm{i}/h)\sinh\tfrac{1}{2}(\beta+(x+1)\pi\mathrm{i}/h). \tag{A.7}$$

Using this relation, we can show that

$$\zeta_{ad}(\beta+\pi\mathrm{i})\zeta_{ad}(\beta) = \varphi_a^{-1}(\beta+d\pi\mathrm{i}/h)\varphi_a^{-1}(\beta+\pi\mathrm{i}-d\pi\mathrm{i}/h)$$
$$= \varphi_d^{-1}(\beta+a\pi\mathrm{i}/h)\varphi_d^{-1}(\beta+\pi\mathrm{i}-a\pi\mathrm{i}/h). \tag{A.8}$$

It holds that

$$\frac{\varphi_d(\beta+\pi\mathrm{i}+a\pi\mathrm{i}/h)\varphi_d(\beta-a\pi\mathrm{i}/h)}{\varphi_d(\beta+\pi\mathrm{i}-a\pi\mathrm{i}/h)\varphi_d(\beta+a\pi\mathrm{i}/h)} = S_{ad}(\beta). \tag{A.9}$$

Making use of these relations, we can show that the function (2.52) satisfies the kinematical residue equations.

(iv) Bound state residue equation: in order to verify that (2.52) satisfies the bound state residue equation for the minimal fusion process $a\times b\to(a+b)$ $(a+b\leqslant N)$, we need the following relations:

$$-\mathrm{i}\lim_{\epsilon\to0}\epsilon\tilde{f}_{\lambda;abd_1\ldots d_n}(\beta+b\pi\mathrm{i}/h+\epsilon,\beta-a\pi\mathrm{i}/h,\beta_1,\ldots,\beta_n)$$
$$= (-1)^n\mathrm{i}^{a-b+1}\mu_a\mu_b\tilde{f}_{\lambda;(a+b)d_1\ldots d_n}(\beta,\beta_1,\ldots,\beta_n)\prod_{j=1}^{n}\varphi_{d_j}(\beta-\beta_j+(b-a)\pi\mathrm{i}/h) \tag{A.10}$$

where

$$\mu_a = \mathrm{i}^{-a}\oint\frac{\mathrm{d}\alpha}{2\pi\mathrm{i}}\varphi_a(\alpha-a\pi\mathrm{i}/h) = \frac{\prod_{j=1}^{a-1}\cos(j\pi/h)}{\prod_{j=1}^{a}\sin(j\pi/h)}. \tag{A.11}$$

It holds that

$$\oint\frac{\mathrm{d}\alpha}{2\pi\mathrm{i}}\varphi_{(a+b)}(\alpha) = \mathrm{i}^{a-b}\mu_a\mu_b = \oint\frac{\mathrm{d}\alpha}{2\pi\mathrm{i}}\varphi_a(\alpha-a\pi\mathrm{i}/h)\oint\frac{\mathrm{d}\alpha'}{2\pi\mathrm{i}}\varphi_b(\alpha'+b\pi\mathrm{i}/h). \tag{A.12}$$

The function $\zeta_{ab}(\beta)$ satisfies a bootstrap equation:

$$\zeta_{ad}(\beta+b\pi\mathrm{i}/h)\zeta_{bd}(\beta-a\pi\mathrm{i}/h) = -\varphi_d^{-1}(\beta+(b-a)\pi\mathrm{i}/h)\zeta_{(a+b)d}(\beta). \tag{A.13}$$

There is a relation among constants:

$$\frac{F_{(a+b)(a+b)}^{(\min)}(\pi\mathrm{i})}{F_{aa}^{(\min)}(\pi\mathrm{i})F_{bb}^{(\min)}(\pi\mathrm{i})}(F_{ab}^{(\min)}((a+b)\pi\mathrm{i}/h))^2$$
$$= \prod_{\substack{x=2max(a,b)+1\\ \text{step }2}}^{2a+2b-1}\sin^2\frac{(x-1)\pi}{2h}\sin^2\frac{(x+1)\pi}{2h} \tag{A.14}$$
$$\frac{2\prod_{j=\min(a,b)}^{max(a,b)}\sin(j\pi/h)}{\prod_{j=1}^{a+b-1}\cos(j\pi/h)}\frac{\nu_a\nu_b}{\nu_{(a+b)}}\mu_a\mu_b F_{ab}^{(\min)}((a+b)\pi\mathrm{i}/h) = \Gamma_{ab}^{(a+b)}.$$

With the aid of these relations, one can prove that the function (2.52) satisfies bound state residue equations.

(v) Cluster properties: finally we analyse cluster properties (2.9).

For $\beta\to\pm\infty$, the building block of the minimal two-body form factor behaves as

$$F_x^{(\min)}(\beta) = -\tfrac{1}{4}\mathrm{e}^{|\beta|}+\cdots. \tag{A.15}$$

Therefore, for $\beta \to \infty$,

$$\zeta_{ab}(\beta) = (-1)^{a+1}\tfrac{1}{2}\mathrm{e}^{\beta} + \cdots \tag{A.16}$$

and for $\beta \to -\infty$,

$$\zeta_{ab}(\beta) = (-1)^{b}\tfrac{1}{2}\mathrm{e}^{-\beta} + \cdots. \tag{A.17}$$

Let us consider the large Λ limit of

$$M^{\lambda}_{a_1\ldots a_m a_{m+1}\ldots a_{m+n}}(\mathrm{e}^{\Lambda}x_1,\ldots,\mathrm{e}^{\Lambda}x_m,x_{m+1},\ldots,x_{m+n}). \tag{A.18}$$

Similar to the case of the ordinary elementary symmetric polynomials [17, 18], the leading behaviour of the generalized symmetric polynomials is determined by the highest degree term.

If $k \leqslant (h-2)m$,

$$E_{a_1\ldots a_m a_{m+1}\ldots a_{m+n};k}(\mathrm{e}^{\Lambda}x_1,\ldots,\mathrm{e}^{\Lambda}x_m,x_{m+1},\ldots,x_{m+n}) \sim \mathrm{e}^{k\Lambda}E_{a_1\ldots a_m;k}(x_1,\ldots,x_m). \tag{A.19}$$

If $k > (h-2)m$,

$$\begin{aligned} E_{a_1\ldots a_m a_{m+1}\ldots a_{m+n};k}&(\mathrm{e}^{\Lambda}x_1,\ldots,\mathrm{e}^{\Lambda}x_m,x_{m+1},\ldots,x_{m+n}) \sim \mathrm{e}^{(h-2)m\Lambda} \\ &\times E_{a_1\ldots a_m;(h-2)m}(x_1,\ldots,x_m)E_{a_{m+1}\ldots a_{m+n};k-(h-2)m}(x_{m+1},\ldots,x_{m+n}) \\ &= = \mathrm{e}^{(h-2)m\Lambda}\left(\prod_{j=1}^{m}x_j^{h-2}\right)E_{a_{m+1}\ldots a_{m+n};k-(h-2)m}(x_{m+1},\ldots,x_{m+n}). \end{aligned} \tag{A.20}$$

Then

$$\begin{aligned} \det(M^{\lambda}_{a_1\ldots a_m a_{m+1}\ldots a_{m+n}}&(\mathrm{e}^{\Lambda}x_1,\ldots,\mathrm{e}^{\Lambda}x_m,x_{m+1},\ldots,x_{m+n})) \\ &= \det(E_{a_1\ldots a_m a_{m+1}\ldots a_{m+n};hi-2j+\lambda-h+1}(\mathrm{e}^{\Lambda}x_1,\ldots,\mathrm{e}^{\Lambda}x_m,x_{m+1},\ldots,x_{m+n}))_{1\leqslant i,j\leqslant m+n} \\ &\quad\sim \det(E_{a_1\ldots a_m a_{m+1}\ldots a_{m+n};hi-2j+\lambda-h+1}(\mathrm{e}^{\Lambda}x_1,\ldots,\mathrm{e}^{\Lambda}x_m))_{1\leqslant i,j\leqslant m} \\ &\quad\times \mathrm{e}^{(h-2)mn\Lambda}\left(\prod_{j=1}^{m}x_j^{(h-2)n}\right) \\ &\quad\times \det(E_{a_{m+1}\ldots a_{m+n};h(i-m)-2(j-m)+\lambda-h+1}(x_{m+1},\ldots,x_{m+n}))_{m+1\leqslant i,j\leqslant m+n} \\ &= \exp\left[\left((h-2)(n+\frac{m-1}{2})+\lambda\right)m\Lambda\right]\left(\prod_{j=1}^{m}x_j^{(h-2)n}\right) \\ &\quad\times \det(M^{\lambda}_{a_1\ldots a_m}(x_1,\ldots,x_m))\det(M^{\lambda}_{a_{m+1}\ldots a_{m+n}}(x_{m+1},\ldots,x_{m+n})). \end{aligned} \tag{A.21}$$

These results allow us to verify that the functions (2.52) satisfy cluster equation with normalization $\langle\phi_{1,1+s}\rangle = 1$.

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